Aerodynamics of Lift Generation

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2025 Graduate Modelling Camp
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February 2, 2025

Outline

Problem Statement

- What are we doing ? The Aerodynamics of lift generation.
- Why are we doing it ? The generation of lift is important in the aircraft industry, wind turbines and in nature with birds and insects.
- How are we going to do it? Firstly we going to consider the thin aerofoil. Discuss circulation in detail, establishing a connection between the vortex and circulation.
- Understanding the mechanism of how mosquitoes generate lift.

Background

In potential flow theory, the flow around an aerofoil is considered irrotational and incompressible. The velocity potential ϕ and the stream function ψ are used to describe the flow.

$$\overline{v} = grad\phi \tag{1}$$

$$V_x = \frac{\partial \phi}{\partial x}, \;\; V_y = \frac{\partial \phi}{\partial y}. \tag{2}$$

- $div\underline{v} = 0$, then there exist a function $\psi(x, y)$ such that

$$V_x = \frac{\partial \psi}{\partial y}, \qquad V_y = -\frac{\partial \psi}{\partial x} \tag{3}$$

Background

- Then

$$w(z) = \phi(x, y) + i\psi(x, y) \quad (z = x + iy) \tag{4}$$

satisfies the Cauchy - Riemann equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{5}$$

and is differentiable in the complex plane. The function $w(z)$ is called the complex potential.

Circulation

- Let C be a closed curve lying in the fluid region. The circulation Γ round C is defined by

$$\Gamma = \oint_C \underline{v} \cdot d\underline{s}, \tag{6}$$

By Stoke's Theorem

$$\Gamma = \int_S (curl \underline{v}) \cdot \underline{n} dS \tag{7}$$

provided the region S spanned by the closed curve C lies entirely in the region of the fluid flow.

Circulation

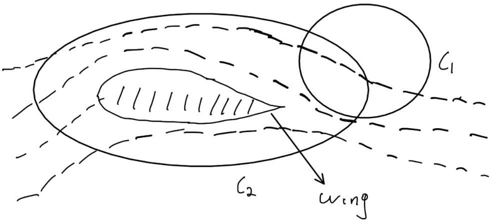


Figure: Two-dimensional irrotational flow

- For any closed curve, C_1, that does not enclose the wing, $\Gamma = 0$ since the flow is irrotational.
- For any closed curve, C_2, that encloses the wing, the Stokes theorem does not apply and $\Gamma \neq 0$.

Circulation

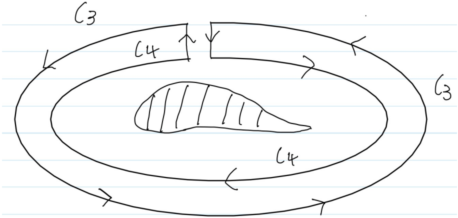


Figure: Circulation round a closed circuit that does not enclose the wing.

$$\Gamma_3 - \Gamma_4 = 0, \quad \Gamma_3 = \Gamma_4 \tag{8}$$

Circulation

- According to Kelvin's theorem, if the fluid is inviscid and incompressible with zero body force then:

$$\frac{D}{Dt}\Gamma = 0 \tag{9}$$

$$\Gamma_1 = 0, \quad \underline{V} = \underline{0}$$

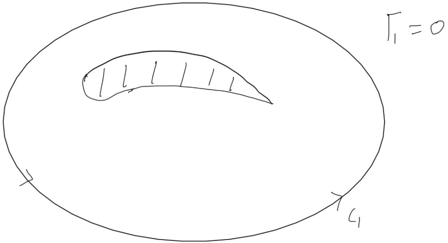


Figure: Aerofoil in a fluid at rest

Circulation

$$\Gamma_1 = \oint_{c_1} \underline{v} \cdot d\underline{s} = 0 \quad \textit{since} \quad \underline{v} = \underline{0}. \tag{10}$$

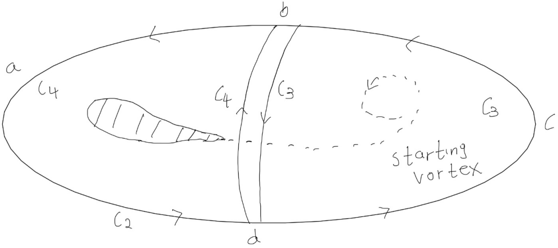


Figure: Formation of Starting Vortex

Circulation

- The whole curve is C_2 and it consists of the same fluid particles as C_1. Since, $\frac{D\Gamma}{Dt} = 0$.
- then $\Gamma_1 = \Gamma_2 = 0, \qquad (\Gamma_1 = 0)$
- Since the cut is common to c_3 and c_4, and transvered in the opposite direction, $\Gamma_3 + \Gamma_4 = 0$, Hence $\Gamma_4 = -\Gamma_3$
- The starting vortex is anticlockwise, hence $\Gamma_3 > 0$. Thus, $\Gamma_4 < 0$. The circulation around the aerofoil is negative.

Kutta-Joukowski Lift Theorem

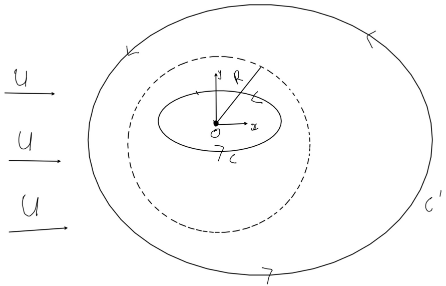


Figure: Two-dimensional body with cross section in closed curve C.

- Consider steady flow past a two-dimensional body with cross-section the closed curve C. The flow is uniform at infinity with speed U in the x direction. The circulation around the body is Γ. Then:

$$F_x = 0, \quad F_y = -\rho U \Gamma \tag{11}$$

Vortex Flows [1]

Flow in which all the streamlines are concentric circles about a given point.

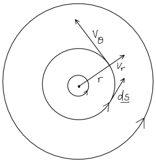


Figure: Vortex flow

- $v_r = 0, \quad v_\theta = \frac{c}{r}, \; v_z = 0.$ (c is a constant)
- For an incompressible fluid (cylindrical polar coordinates) the condition

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial v_z}{\partial z} = 0 \tag{12}$$

  is satisfied.
- The flow is irrotational for $r \neq 0$.

Vortex Flows : Incompressible Irrotational Flow

- Circulation around a given streamline of radius r

$$\begin{aligned} \Gamma &= \oint_c \underline{v} \cdot d\underline{s} \\ &= \int_0^{2\pi} \frac{c}{r}\ r\ d\theta \\ &= 2\pi c. \end{aligned} \tag{13}$$

- Thus

$$c = \frac{\Gamma}{2\pi}. \tag{14}$$

It follows that

$$v_r = 0, \quad v_\theta = \frac{\Gamma}{2\pi r}, \quad v_z = 0. \tag{15}$$

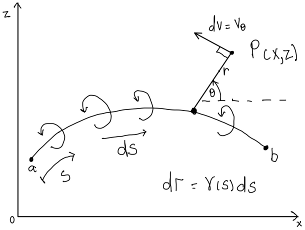


Figure: Edge view of vortex sheet

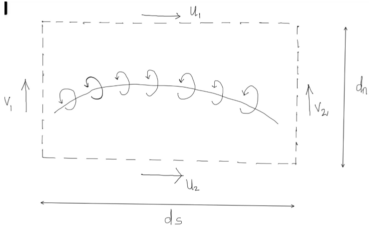
The small section of vortex sheet of strength γds induces an infinitesimal velocity $dv, (dv = v_\theta)$ at point P. It follows from (15) that:

$$dv = \frac{\gamma(s)ds}{2\pi r} \tag{16}$$

- The circulation around the vortex sheet is thus

$$\Gamma = \int_a^b \gamma(s)ds \tag{17}$$

Considering the vortex sheet



One has

$$\Gamma = \oint_c \underline{v} \cdot d\underline{s}$$

$$\therefore d\Gamma = (u_2 - u_1)ds + (v_2 - v_1)dn. \tag{18}$$

As $dn \to 0$,

$$d\Gamma = (u_2 - u_1)ds \tag{19}$$

since $\Gamma = \gamma(s)ds$,

$$\gamma(s) = (u_2 - u_1).$$

Kutta Condition

Types of trailing edges:

- Trailing edge with a finite angle
- Cusped trailing edge

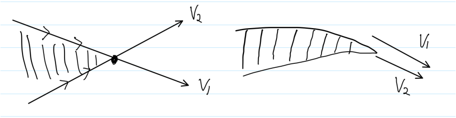


Figure: Trailing edge with finite angle and cusped trailing edge

$$\gamma(s) = V_2 - V_1 \tag{20}$$

Finite angle: $V_1 = V_2 = 0, \gamma(TE) = 0 \quad (TE = trailing\ edge).$
Cusp: $V_1 = V_2 \neq 0, \quad \gamma(TE) = 0.$

Thin Aerofoil Theory

A thin aerofoil is simulated by a vortex sheet placed along the camber line.

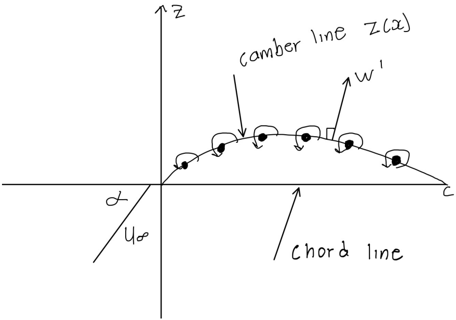


Figure: Vortex sheet on the camber line

The aerofoil is thin . We make the approximation of putting the vortex sheet on the chord line. Thus $\gamma = \gamma(x)$.

- The camber line is a streamline of the flow
- Kutta condition $\gamma(c) = 0$

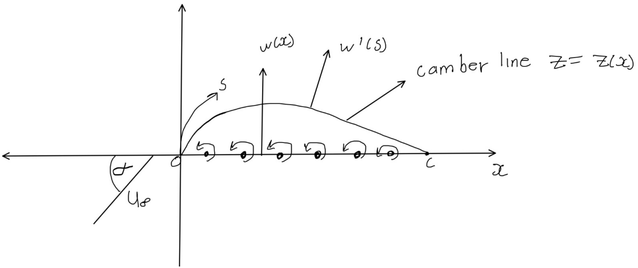


Figure: Vortex sheet on chord line

Thin Aerofoil Theory

Let $u_{\infty n}$ be the component of freestream velocity normal to the camber line. It can be shown that approximately

$$u_{\infty n} = U_{\infty}(\alpha - \frac{dz}{dx}) \tag{21}$$

Thin Aerofoil

Let

w'(x) = component of velocity normal to camber line induced by the vortex sheet

w(x) = component of velocity normal to chord line induced by the vortex sheet

For a thin aerofoil,

$$w'(s) = w(x)$$

For an infinitesimal vortex of strength $\gamma(\xi)d\xi$ located ξ away from the origin along chord line:

$$dw = \frac{\gamma(\xi)d\xi}{2\pi(x-\xi)} \tag{22}$$

$$\tag{23}$$

$$\therefore w(x) = \frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{x-\xi} \tag{24}$$

Thin Aerofoil

For camber line to be a streamline:

$$\underbrace{u_{\infty n} + w'(s)} = 0$$

But $w'(s) = w(x)$ (thin aerofoil). Thus

$$u_{\infty n} = -w(x)$$

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{\xi - x} = (\alpha - \frac{dz}{dx})U_\infty \tag{25}$$

Summary for Thin Aerofoils

- In this work, we will be solving the aerofoil equation

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{\xi - x} = \left(\alpha - \frac{dz}{dx}\right)U_\infty \tag{26}$$

subject to the Kutta condition

$$\gamma(c) = 0 \tag{27}$$

- We derive the circulation and lift from the solution to aerofoil equation:

$$\Gamma = \int_0^c \gamma(x)dx \tag{28}$$

$$L = -\rho U_\infty \Gamma \tag{29}$$

Aerofoil

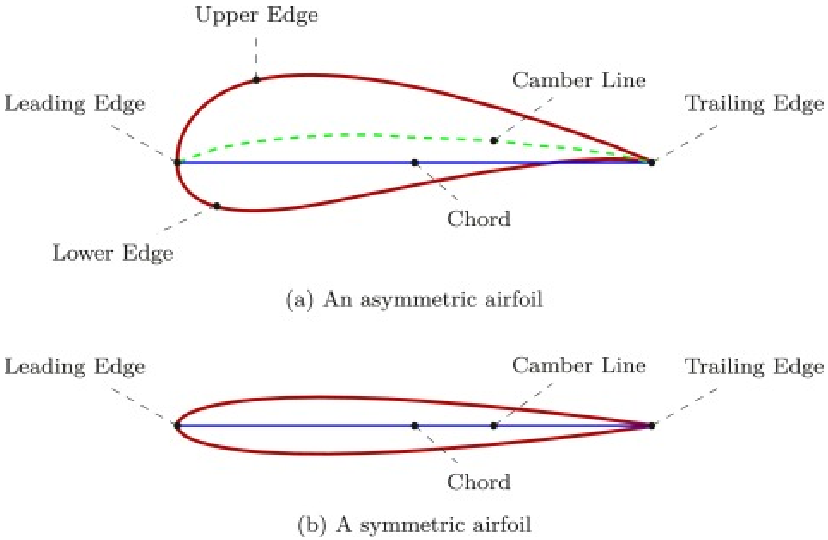


Figure: Representing the Aerofoils

Asymmetrical Aerofoil

- For the asymmetrical aerofoil,

$$z(x) = \varepsilon x(c - x). \tag{30}$$

Thus

$$\frac{dz}{dx} = \varepsilon(c - 2x), \tag{31}$$

and the aerofoil model reduces to

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{\xi - x} = (\alpha - \varepsilon(c - 2x))U_\infty, \tag{32}$$

$$\gamma(c) = 0, \tag{33}$$

$$\Gamma = \int_0^c \gamma(x)dx \tag{34}$$

$$L = -\rho U_\infty \Gamma \tag{35}$$

Asymmetrical Aerofoil

- Let

$$\xi^* = \xi - \frac{c}{2}, \quad \text{and} \quad x^* = x - \frac{c}{2}. \tag{36}$$

- The aerofoil equation becomes

$$\frac{1}{2\pi}\int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{\gamma(\xi^*)d\xi^*}{\xi^* - x^*} = (\alpha + 2\varepsilon x^*)U_\infty. \tag{37}$$

- Further we let

$$\xi^* = \frac{c}{2}cos\theta, \quad \text{and} \quad x^* = \frac{c}{2}cos\phi. \tag{38}$$

- Now,the aerofoil equation becomes

$$\frac{1}{2\pi}\int_{0}^{\pi} \frac{\gamma(\theta)sin\theta d\theta}{cos\theta - cos\phi} = (\alpha + \varepsilon c\cos\phi)U_\infty. \tag{39}$$

We make an expansion of the form

$$\gamma(\theta) = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\gamma_n \cos n\theta, \tag{40}$$

where the $\gamma_n' s$ are constants. Equation (39) becomes

$$\frac{1}{2\pi}\sum_{n=0}^{\infty}\gamma_n \int_0^{\pi} \frac{\cos n\theta d\theta}{cos\theta - cos\phi} = (\alpha + \varepsilon c \cos\phi)U_\infty. \tag{41}$$

Asymmetrical Aerofoil

- We prove later that,

$$\int_0^{\pi} \frac{\cos n\theta d\theta}{cos\theta - cos\phi} = \frac{\pi \sin n\phi}{\sin \phi}. \tag{42}$$

Thus, equation (41) becomes

$$\sum_{n=1}^{\infty} \gamma_n \sin n\phi = (2\alpha \sin \phi + \varepsilon c \sin 2\phi) U_\infty. \tag{43}$$

- We equate the coefficiences of $\sin n\phi$ for $n \geq 1$.

$$\begin{aligned} \sin \phi : \gamma_1 &= 2\alpha U_\infty \\ \sin 2\phi : \gamma_2 &= \varepsilon c U_\infty \\ \sin n\phi,\ \ n \geq 3 : \gamma_n &= 0. \end{aligned}$$

Thus

$$\gamma(\phi) = \frac{1}{\sin\phi}[\gamma_0 + 2\alpha U_\infty \cos\phi + \varepsilon c U_\infty \cos 2\phi]. \quad (44)$$

For a finite solution as ϕ approaches zero. we require the:

$$\gamma_0 + 2\alpha U_\infty + \varepsilon c U_\infty = 0. \quad (45)$$

$$(46)$$

Thus

$$\gamma(\phi) = -\frac{U_\infty}{\sin\phi}[2\alpha(1-\cos\phi) + \varepsilon c(1-\cos 2\phi)]. \quad (47)$$

By L'Hopital's rule , the Kutta condition $\gamma(0) = 0$ is identically satisfied.

Asymmetrical Aerofoil

Substituting $\gamma(\phi)$ into

$$\Gamma = \int_0^c \gamma(\theta) sin(\theta) d\theta$$

we obtain for the circulation

$$\Gamma = -c\pi U_\infty(\alpha + \frac{\varepsilon c}{2}). \tag{48}$$

Thus the lift is

$$L = -\rho U_\infty \Gamma \tag{49}$$

$$= \pi\rho c U_\infty^2(\alpha + \frac{\varepsilon c}{2}). \tag{50}$$

Symmetrical Aerofoil

For the symmetrical aerofoil, $z(x) = 0$. Thus $\frac{dz}{dx} = 0$, and the aerofoil model reduces to

$$\frac{1}{2\pi}\int_0^c \frac{\gamma(\xi)d\xi}{\xi - x} = \alpha U_\infty \tag{51}$$

$$\gamma(c) = 0 \tag{52}$$

$$\Gamma = \int_0^c \gamma(x)dx \tag{53}$$

$$L = -\rho U_\infty \Gamma \tag{54}$$

The solution is obtained by putting $\varepsilon = 0$ in (48) and (50)

$$\Gamma = \pi c U_\infty \alpha$$

and

$$L = \pi c \rho U_\infty^2 \alpha$$

Camber increases lift.

Evaluation of Integral

$$\int_0^\pi \frac{\cos n\theta d\theta}{cos\theta - cos\phi} = \frac{\pi \sin n\phi}{\sin \phi} \tag{55}$$

Let

$$I_m(\phi) = \int_0^\pi \frac{\cos m\theta d\theta}{\cos\theta - \cos\phi} \tag{56}$$

It follows for $m \geq 1$

$$I_{m+1}(\phi) + I_{m-1}(\phi) = \int_0^\pi \frac{\cos(m+1)\theta d\theta}{\cos\theta - \cos\phi} + \int_0^\pi \frac{\cos(m-1)\theta d\theta}{\cos\theta - \cos\phi}. \tag{57}$$

But

$$cosA + cosB = 2cos\frac{1}{2}(A+B)cos\frac{1}{2}(A-B).$$

Hence

$$I_{m+1}(\phi) + I_{m-1}(\phi) = 2\int_0^\phi \frac{cosm\theta cos\theta}{cos\theta - cos\phi} d\theta$$

$$I_{m+1}(\phi) + I_{m-1}(\phi) = 2\int_0^\pi cosm\theta d\theta \quad + 2cos\phi I_m(\phi)$$

and therefore

$$I_{m+1}(\phi) - 2cos\phi I_m(\phi) + I_{m-1}(\phi) = 0, \quad m \geq 1. \tag{58}$$

Difference equation for $I_m(\phi)$.
Initial conditions:

$$m = 0 : \quad I_0(\phi) = \int_0^\pi \frac{d\theta}{cos\theta - cos\phi} = 0$$

Integrated by making transformation $t = tan\frac{\theta}{2}$.

$$m = 1 : \quad I_1(\phi) = \int_0^\pi \frac{cos\theta d\theta}{cod\theta - cos\phi} = \int_0^\pi d\theta + cos\phi \int_0^\pi \frac{d\theta}{cos\theta - cos\phi} = \pi$$

The Important result

$$Let: \quad I_m(\phi) = A\lambda^m, \quad A = constant$$
$$\lambda^2 - 2cos\phi\lambda + 1 = 0$$
$$\lambda_1 = e^{i\phi}, \quad \lambda_2 = e^{-i\phi}$$

Thus

$$I_m = A_1 e^{im\phi} + A_2 e^{-im\phi}$$
$$= c_1 cos(m\phi) + ic_2 sin(m\phi) \quad (c_1 = A_1 + A_2, \quad c_2 = A_1 - A_2)$$

Initial conditions:

$$I_0 = 0, \quad I_1 = \pi$$
$$c_1 = 0, \quad c_2 = -\frac{i\pi}{sin\phi}$$

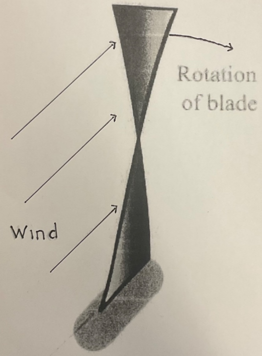
hence

$$I_m(\phi) = \frac{\pi sinm\phi}{sin\phi}$$

Important Result

$$\int_0^{\pi} \frac{cosm\theta d\theta}{cos\theta - cos\phi} = \frac{\pi sinm\phi}{sin\phi} \tag{59}$$

Rotation of Wind Turbine blades



(a) Angle of twist

Rotation of Wind Turbine blades : Example

Blade of length 80m makes one full rotation in 10s. The wind speed is 10m/s.

$$\text{The angular velocity} = \frac{2\pi}{10}\text{radians per second} \tag{60}$$

$$\text{Tip speed velocity} = 16\pi m/s = 500m/s \tag{61}$$

Aerofoil dynamics of Mosquito Lift Generation

- Leading edge vortices are the primary mechanism lift generation for small insects, including mosquitoes.
- This mechanism is critical for hovering flight, enabling mosquitoes to remain stationary
- Trailing edge vortices are equally essential for maintaining aerodynamic balance and energy efficiency during flight
- During the upstroke, the trailing edge plays a role in capturing energy from the wake left behind by the previous downstroke. This wake capture mechanism enhances flight efficiency.
- Rapid rotations of the wings enhance lift by creating an additional circulation.
- The mosquitoes have the longest wings of all the insects for their body size.
- The mosquitoes flap their wings with a frequency up to 800Hz

Conclusions

- The aerofoil model was derived.
- The model was solved for both a symmetrical and an asymmetrical aerofoil.
- It was observed that whenever the circulation is negative then the lift is positive.
- The greater the wind speed, the greater the magnitude of the lift.
- The greater the air density, the greater the magnitude of the lift.
- Without changing the angle of attack the effect of a non-zero camber line is to increase the lift.

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Questions?